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Ricci Semi-symmetric Hypersurfaces in Complex Two-Plane Grassmannians

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Abstract In this paper, we considered Ricci semi-symmetric real hypersurface in complex two-plane Grassmannians. Then we prove the non-existence of Ricci semi-symmetric Hopf hypersurfaces in complex two-plane Grassmannians by using the method of simultaneous diagonalization for pairwise commutative matrices.

Keywords Real hypersurfaces · Hopf hypersurface · Complex two-plane Grassmannians · Ricci semi-symmetric · Symmetric operator · Simultaneous diagonalization

Mathematics Subject Classification Primary 53C40; Secondary 53C15

Introduction

The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is a kind of Hermitian symmetric space of compact irreducible type with rank 2. Remarkably, the manifolds are equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ satisfying $JJ_\nu = J_\nu J$ $(\nu = 1, 2, 3)$ where $\{J_\nu\}_{\nu=1,2,3}$ is an orthonormal basis of $\mathfrak{J}$. When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism

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$\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann Manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper we always assume $m \geq 3$ (see [2]).

Suppose M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$. Let N be a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Since $G_2(\mathbb{C}^{m+2})$ has the Kähler structure J, we may define the *Reeb vector field* $\xi = -JN$ and a one-dimensional distribution $[\xi] = \mathcal{C}^\perp$ where $\mathcal{C}$ denotes the orthogonal complement in T_xM, $x \in M$, of the Reeb vector field ξ. The Reeb vector field ξ is said to be *Hopf* if $\mathcal{C}$ (or $\mathcal{C}^\perp$) is invariant under the shape operator A of M. The one-dimensional foliation of M defined by the integral curves of ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* if and only if the Hopf foliation of M is totally geodesic. By the formulas in [7, Sect. 2], it can be checked that ξ is Hopf vector field if and only if M is Hopf hypersurface.

From the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$, there naturally exists *almost contact 3-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Put $\mathcal{Q}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent bundle TM of M. In addition, denoted by $\mathcal{Q}$ the orthogonal complement of $\mathcal{Q}^\perp$ in TM. It is the quaternionic maximal subbundle of TM. Thus, the tangent bundle of $M$is expressed by a direct sum of $\mathcal{Q}$ and $\mathcal{Q}^\perp$.

For two distributions $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ defined above, we may consider two natural invariant geometric properties under the shape operator A of M, that is, $A\mathcal{C}^\perp \subset \mathcal{C}^\perp$ and $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$. By using the result of Alekseevskii [1], Berndt and Suh [2, Theorem 1] have classified all real hypersurfaces with two natural invariant properties in $G_2(\mathbb{C}^{m+2})$ as follows:

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{Q}^\perp$ are invariant under the shape operator of M if and only if

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

In the case (A), we say M is of Type (A). Similarly in the case (B) we say M is of Type (B).

Regarding the parallelism of $(1, 1)$-type tensor field T, (i.e., $\nabla T = 0$) on real hypersurface M in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, there are many well-known results. Many geometers have verified non-existence properties and some characterizations which show many kinds of parallelisms, such as parallel, Reeb parallel, or generalized Tanaka-Webster parallel (see [13,14,16] and [17]).

Recently, Panagiotidou and Tripathi [10] considered the notion of real hypersurfaces with *semi-parallel normal Jacobi operator* $\bar{R}_N$ in $G_2(\mathbb{C}^{m+2})$, that is, $R(X, Y) \cdot \bar{R}_N = 0$. Motivated by this, we want to study the semi-parallelism on Ricci tensor. The Ricci tensor S on real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is defined by

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \ldots, e_{4m-1}\}$ is an orthonormal basis of the tangent space T_xM, $x \in M$ in $G_2(\mathbb{C}^{m+2})$ and $X, Y \in T_xM$ (see [15]). Hereafter, we consider that X and Y are all tangent vector fields on M. A Riemannian manifold is called *Ricci semi-symmetric* if

$$R(X,Y)\cdot S=0, \tag{*}$$

where R is the curvature tensor of type (1,3) and $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space (see [5]).
In this paper, we consider Ricci semi-symmetric Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$.

By [2, Theorem 1] and *that of simultaneous diagonalizable matrices* in [3], we prove the non-existence of Ricci semi-symmetric Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ as follows:

Theorem *There does not exist a Ricci semi-symmetric Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.*

Since semi-parallelism, that is, $R(X,Y)\cdot S=0$ is weaker than parallel Ricci tensor, i.e., $\nabla S=0$ (see [16]), by our Theorem mentioned above we obtain the following result

Corollary 1 *There does not exist a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel Ricci tensor.*

In [18], the Ricci tensor S for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be recurrent if $(\nabla_X S)Y=\omega(X)SY$, where ω is a one form defined on M in $G_2(\mathbb{C}^{m+2})$. From [9, Theorem 20] and our Theorem, we also get another corollary as follows:

Corollary 2 *There does not exist a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with recurrent Ricci tensor.*

In order to prove our main result, the paper is organized as follows. In Sect. 1 we recall some fundamental formulas including the Gauss equation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. In Sect. 2 we prove that the Reeb vector field ξ of a Ricci semi-symmetric Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$. Some lemmas for proving commuting conditions between symmetric operators are given. In Sect. 3, we show that a Ricci semi-symmetric Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfies $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$ and check a non-existence property for real hypersurface in $G_2(\mathbb{C}^{m+2})$ with given conditions.

1 Preliminaries

In this paper, suppose M is a real hypersurface of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, that is, a submanifold of codimension 1 in $G_2(\mathbb{C}^{m+2})$. Let us denote by R the Riemannian curvature and $\bar{R}$ the Riemannian curvature tensor on $G_2(\mathbb{C}^{m+2})$, respectively. That is, $R=\bar{R}|_M$ tensor on M. Hereafter unless otherwise stated, X, Y, Z, and W are tangent vector fields on M. In this section, we recall some basic formulas and the Gauss equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [4,7,12,15]). The induced Riemannian metric on M (resp., $G_2(\mathbb{C}^{m+2})$) is denoted by g (resp., $\bar{g}$). Let ∇ and $\bar{\nabla}$ be the Riemannian connections of (M,g) and $(G_2(\mathbb{C}^{m+2}),\bar{g})$, respectively. Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. J (resp., $\mathfrak{J}$=$Span\{J_\nu\}_{\nu=1,2,3}$) denotes the Kähler structure (resp., the quaternionic Kähler structure). We put

$$JX = \phi X + \eta(X)N \text{ and } J_\nu X = \phi_\nu X + \eta_\nu(X)N,$$

where ϕX (resp., $\phi_\nu X$) is the tangential part of JX (resp., $J_\nu X$), and $\eta(X) = g(X, \xi)$ (resp., $\eta_\nu(X) = g(X, \xi_\nu)$) is the coefficient of the normal part of JX (resp., $J_\nu X$). In this case, we call ϕ the structure tensor field of M.

The Gauss equation is given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY \\ & + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\Big\} \\ & - \sum_{\nu=1}^{3}\Big\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\Big\} \\ & - \sum_{\nu=1}^{3}\Big\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\Big\}\xi_\nu. \end{aligned} \tag{1.1}$$

From the definition of the Ricci tensor S and by the fundamental formulas in [15, Sect. 2], we have

$$\begin{aligned} SX = {} & (4m+7)X - 3\eta(X)\xi + hAX - A^2X \\ & + \sum_{\nu=1}^{3}\{-3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X - \eta_\nu(\phi X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\}, \end{aligned} \tag{1.2}$$

where h denotes the trace of the shape operator A in M with respect to N.

The structure Jacobi operator R_ξ is defined by [8, Sect. 1]

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ &\quad + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\Big\} + \eta(A\xi)AX - \eta(AX)A\xi. \end{aligned} \tag{1.3}$$

[6, Lemma A] If M is a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, then we have the following two equations:

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y), \tag{1.4}$$

and

$$
\begin{aligned}
\alpha A\phi X + \alpha\phi AX - 2A\phi AX + 2\phi X = 2\sum_{\nu=1}^{3}\Big\{ & -\eta_\nu(X)\phi\xi_\nu - \eta_\nu(\phi X)\xi_\nu \\
& - \eta_\nu(\xi)\phi_\nu X + 2\eta(X)\eta_\nu(\xi)\phi\xi_\nu + 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\}, \qquad (1.5)
\end{aligned}
$$

where the Reeb function $\alpha = \eta(A\xi)$ on M.

2 A Key Lemma

We first give the fundamental equation for a Ricci semi-symmetric real hypersurface M in $G_2(\mathbb{C}^{m+2})$. A real hypersurface M is called *Ricci semi-symmetric* if $R(X, Y)\cdot S = 0$, that is, $\big(R(X, Y)S\big)Z = 0$ for any vector field X, Y, and Z. It is equivalent to

$$
R(X, Y)(SZ) = S(R(X, Y)Z). \qquad (2.1)
$$

Since the Ricci tensor S is symmetric, we have

$$
R(SX, Y)Z = R(X, SY)Z. \qquad (**)
$$

In order to prove our Theorem, let us show that the Reeb vector field ξ belongs to either or the distribution $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ or its orthogonal complement the distribution $\mathcal{Q}$ with the assumption of Ricci semi-symmetric as follows:

Lemma 2.1 *Let M be a Ricci semi-symmetric Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.*

Proof We consider that the Reeb vector fields ξ satisfies

$$
\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1
$$

for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^\perp$, and $\eta(X_0)\eta(\xi_1) \neq 0$. Let $A\xi = \alpha\xi$. In the case of $\alpha = 0$, by (1.4), ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$ which contradicts the assumption (see [12]). If $\alpha \neq 0$, from (1.2) (resp., (1.3)), we have

$$
S\xi = \sigma_0\xi - 4\eta_1(\xi)\xi_1, \quad \text{where} \quad \sigma_0 := 4m + 4 + h\alpha - \alpha^2, \qquad (2.2)
$$

$$
R_\xi(\xi_1) = \alpha A\xi_1 - \alpha^2\eta_1(\xi)\xi. \qquad (2.3)
$$

Substituting $X = Y = Z = \xi$ into (**), we get $R(S\xi, \xi)\xi = R(\xi, S\xi)\xi$, which means $R_\xi(S\xi) = 0$. Since $-4\eta_1(\xi) \neq 0$ and (1.3), we have $R_\xi(\xi_1) = 0$. From (2.3), we obtain $A\xi_1 = \alpha\eta_1(\xi)\xi$ and $AX_0 = \alpha\eta(X_0)\xi$.

By putting $X = X_0$ into (1.5), we have

$$
A\phi X_0 = \sigma_1\phi X_0, \quad \text{where} \quad \sigma_1 := \frac{-4\eta^2(X_0)}{\alpha}. \qquad (2.4)
$$

By using (2.4) and substituting $X = \phi X_0$ into (1.2) (resp., (1.3)), we obtain

$$S\phi X_0 = \sigma_2 \phi X_0, \quad \text{where} \quad \sigma_2 = 4m + 8 + h\sigma_1 - \sigma_1^2, \tag{2.5}$$

$$R_\xi(\phi X_0) = 0. \tag{2.6}$$

By substituting $X = \phi X_0$, $Y = \xi$, $X = \xi$ into (**) and using (2.2), (2.5), (2.6), we have $0 = -4\eta_1(\xi)R(\phi X_0, \xi_1)\xi$.

Since we assumed $\eta(X_0)\eta(\xi_1) \neq 0$, this equation becomes

$$0 = R(\phi X_0, \xi_1)\xi. \tag{2.7}$$

Putting $X = \phi X_0$, $Y = \xi_1$, and $Z = \xi$ into (1.1) and using (2.4), (2.7) becomes

$$\begin{aligned} 0 &= R(\phi X_0, \xi_1)\xi \\ &= \eta_1(\xi)\phi X_0 + g(\phi_1^2 X_0, \xi)\phi_1\phi^2 X_0 - \eta_1^2(\xi_1)\phi_1\phi^2 X_0 + \alpha\eta_1(\xi)A\phi X_0 \\ &= -4\eta_1(\xi)\eta^2(X_0)\phi X_0. \end{aligned}$$

This means $\phi X_0 = 0$. However $g(\phi X_0, \phi X_0) = 1 - \eta^2(X_0) = \eta^2(\xi_1)$ never vanishes, it is a contradiction. Accordingly, the lemma is proved. □

Next we further study the case $\xi \in \mathcal{Q}^\perp$.

Lemma 2.2 [3] *If A,B,C are diagonalizable matrices and commute with each other, then there exists a basis* $\{e_k\}_{k=1}^{4m-1}$ *which simultaneously diagonalizes A,B,C.*

Lemma 2.3 [11] *Let M be a Hopf hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. *If the Reeb vector field* ξ *belongs to the distribution* $\mathcal{Q}^\perp$, *then* $SA = AS$.

On the other hand, if $\xi = \xi_1 \in \mathcal{Q}^\perp$, (1.3) is reduced to

$$\begin{aligned} R_\xi(X) = X - \eta(X)\xi + 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3 \\ - \phi_1\phi X + \eta(A\xi)AX - \eta(AX)A\xi \end{aligned} \tag{2.8}$$

and we also have (see [11])

$$\phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX, \tag{2.9}$$

$$A\phi X = 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X. \tag{2.10}$$

Related to the shape operator A and the structure Jacobi operator R_ξ, we assert the following:

Lemma 2.4 *Let M be a Hopf hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. *If the Reeb vector field* ξ *belongs to the distribution* $\mathcal{Q}^\perp$, *then* $R_\xi A = AR_\xi$.

Proof Applying A (Substituting X as AX) to (2.8) and using (2.9) and (2.10), we have

$$\begin{cases} AR_\xi X = \alpha A^2 X - (\alpha^3 + 2\alpha)\eta(X)\xi + 2AX, \\ R_\xi AX = \alpha A^2 X - (\alpha^3 + 2\alpha)\eta(X)\xi + 2AX. \end{cases}$$

Thus, we have $R_\xi A = AR_\xi$. □

3 Proof of Theorem

In this section, we prove the non-existence of Ricci semi-symmetric Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. For this purpose, we give the following:

Lemma 3.1 *There does not exist any Ricci semi-symmetric Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with ξ belongs to $\mathcal{Q}^\perp$ everywhere.*

Proof Putting $Y = Z = \xi$ into (**) (resp., (2.1)), we have

$$R_\xi(SX) = \sigma R_\xi(X) \tag{3.1}$$
$$SR_\xi(X) = \sigma R_\xi(X), \tag{3.2}$$

where $\sigma = 4m + h\alpha - \alpha^2$. Thus,

$$R_\xi S = SR_\xi. \tag{3.3}$$

By Lemmas 2.2, 2.3, 2.4, and 3.3, we know that there exists an orthonormal basis $\{e_k\}_{k=1}^{4m-1}$ such that

$$Ae_k = \lambda_k e_k, \tag{3.4}$$
$$R_\xi(e_k) = \gamma_k e_k, \tag{3.5}$$
$$Se_k = t_k e_k, \tag{3.6}$$

where $k = 1, 2, \ldots, 4m-1$. Since $R_\xi(\xi) = 0$, there exist $j \in \{1, \ldots, 4m-1\}$, such that $R_\xi(e_j) = 0$.

Thus, the tangent space can be split into $T_x M = \mathfrak{D}_0(x) \oplus \mathfrak{D}_0^\perp(x)$, where $x \in M$ and

$$\begin{cases} \mathfrak{D}_0(x) = \operatorname{span}\Big\{e_j \in \{e_k\}_{k=1}^{4m-1} \mid R_\xi(e_j) = 0\Big\} & \text{at } x, \\ \mathfrak{D}_0^\perp(x) = \operatorname{span}\Big\{e_i \in \{e_k\}_{k=1}^{4m-1} \mid R_\xi(e_i) \neq 0\Big\} & \text{at } x. \end{cases}$$

Since $\xi = \xi_1$, the equation (1.4) is reduced to

$$\begin{aligned} SX = {} & (4m+7)X - 7\eta(X)\xi - 2\eta_2(X)\xi_2 - 2\eta_3(X)\xi_3 \\ & + \phi_1\phi X + hAX - A^2X. \end{aligned} \tag{3.7}$$

By (3.4) and (3.6), putting $X = e_j \in \mathfrak{D}_0(x)$ into (2.8) (resp., (3.7)), we have

$$0 = (1 + \alpha\lambda_j)e_j - \eta(e_j)(\alpha^2 + 1)\xi + 2\eta_2(e_j)\xi_2 + 2\eta_3(e_j)\xi_3 - \phi_1\phi e_j, \tag{3.8}$$

$$t_j e_j = (4m + 7 + h\lambda_j - \lambda_j^2)e_j - 7\eta(e_j)\xi - 2\eta_2(e_j)\xi_2 - 2\eta_3(e_j)\xi_3 + \phi_1\phi e_j. \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$(4m + 8 + h\lambda_j - \lambda_j^2 + \alpha\lambda_j - t_j)e_j = (\alpha^2 + 8)\eta(e_j)\xi. \tag{3.10}$$

Since $R_\xi(X)$ never vanishes for all tangent vectors X belongs to $\mathfrak{D}_1(x)$. Thus, $SR_\xi(X) = \sigma R_\xi(X)$ is equivalent to $SX = \sigma X$ for any $X \in \mathfrak{D}_1(x)$. Since $R_\xi(\xi) = 0$, the Reeb vector field ξ belongs to $\mathfrak{D}_0(x)$, thus $\dim \mathfrak{D}_0(x) \geq 1$.

Now, we may consider the following cases:

Case I $\dim \mathfrak{D}_0(x) = 1$.

In this case, $T_xM = \mathfrak{D}_0(x) \oplus \mathfrak{D}_1(x) = [\xi](x) \oplus \mathfrak{D}_1(x)$. Since $\mathfrak{D}_1(x)$ is ϕ-invariant vector space and $S\xi = \sigma\xi$, we have $SX = \sigma X$ for any tangent vector field X on M. Thus, we have $S\phi = \phi S$. By a result of Suh [15, Theorem]: Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, i.e., $S\phi = \phi S, m \geq 3$. Then M is locally congruent to a real hypersurface of Type (A).

Case II $\dim \mathfrak{D}_0(x) = \ell \geq 2$.

In this case, $T_xM = \mathfrak{D}_0(x) \oplus \mathfrak{D}_1(x)$. Since the Reeb vector field ξ satisfies $A\xi = \alpha\xi$, $R_\xi(\xi) = 0\xi$ and $S\xi = \sigma\xi$, we may put $\mathfrak{D}_0(x) = [\xi](x) \oplus \text{span}\{e_{k_2}, \ldots, e_{k_j}, \ldots, e_{k_\ell}\}$, where $j \geq 2$. Then we have $\eta(e_{k_j}) = 0$ for $2 \leq j \leq l$. Putting $X = e_{k_j}$ into (2.8) and (3.7), we have

$$0 = R_\xi(e_{k_j}) = (1 + \alpha\lambda_{k_j})e_{k_j} + 2\eta_2(e_{k_j})\xi_2 + 2\eta_3(e_{k_j})\xi_3 - \phi_1\phi e_{k_j}, \tag{3.11}$$

$$t_{k_j}e_{k_j} = Se_{k_j} = (4m + 8 + h\lambda_{k_j} - \lambda_{k_j}^2 + \alpha\lambda_{k_j})e_{k_j}. \tag{3.12}$$

If we apply the shape operator A to (3.11), we obtain

$$\begin{aligned} 0 &= (1 + \alpha\lambda_{k_j})Ae_{k_j} + 2\eta_2(e_{k_j})A\xi_2 + 2\eta_3(e_{k_j})A\xi_3 - A\phi_1\phi e_{k_j} \\ &= (2 + \alpha\lambda_{k_j})\lambda_{k_j}e_{k_j}. \end{aligned}$$

So we may consider the following two subcases:

Subcase I $2 + \alpha\lambda_{k_j} = 0$, where $j \geq 2$.

$$2 + \alpha\lambda_{k_j} = 0 \quad \left(i.e., \quad \lambda_{k_j} = -\frac{2}{\alpha}\right). \tag{3.13}$$

Using (3.13), (3.11) is changed into

$$0 = -e_{k_j} + 2\eta_2(e_{k_j})\xi_2 + 2\eta_3(e_{k_j})\xi_3 - \phi_1\phi e_{k_j}. \tag{3.14}$$

Applying ϕ_1 to (3.14), we have

$$\phi_1 e_{k_j} = 2\eta_2(e_{k_j})\xi_3 - 2\eta_3(e_{k_j})\xi_2 + \phi e_{k_j}. \tag{3.15}$$

Since $\xi = \xi_1$, (1.5) is reduced to

$$A\phi AX = \frac{\alpha}{2}A\phi X + \frac{\alpha}{2}\phi AX + \phi X + \phi_1 X - 2\eta_2(X)\xi_3 + 2\eta_3(X)\xi_2. \tag{3.16}$$

By putting $X = e_{k_j}$ into (3.16) and by using (3.13) and (3.15), we get

$$A\phi e_{k_j} = \sigma_3 \phi e_{k_j} \quad \text{where} \quad \sigma_3 = \frac{-2\alpha}{\alpha^2+4}. \tag{3.17}$$

Substituting $X = \phi e_{k_j}$ into (1.2) (resp., (1.3)) and using (3.17), we obtain

$$R_\xi(\phi e_{k_j}) = (\alpha\sigma_3 + 2)\phi e_{k_j}, \tag{3.18}$$

$$S\phi e_{k_j} = (4m + 6 + h\sigma_3 - \sigma_3^2)\phi e_{k_j}. \tag{3.19}$$

By using (3.18) and (3.19) and substituting $X = \phi e_{k_j}$ into (3.1), it follows that

$$(6 + \sigma_3 h - h\alpha - \sigma_3^2 + \alpha^2)(\alpha\sigma_3 + 2)\phi e_{k_j} = 0.$$

Since $\alpha\sigma_3 + 2 = \frac{8}{\alpha^2+8} > 0$ (i.e., $\alpha\sigma_3 + 2$ never vanishes), we have

$$6 + \sigma_3 h - h\alpha - \sigma_3^2 + \alpha^2 = 0, \tag{3.20}$$

$$Se_{k_j} = \left(4m + 6 - h\frac{2}{\alpha} - \left(\frac{2}{\alpha}\right)^2\right)e_{k_j}. \tag{3.21}$$

By using (3.19), (3.21) and putting $X = \phi e_{k_j}$, $Y = e_{k_j}$ into (**), we obtain

$$\left(\sigma_3 + \frac{2}{\alpha}\right)\left(h - \sigma_3 + \frac{2}{\alpha}\right)R(\phi e_{k_j}, e_{k_j})Z = 0. \tag{3.22}$$

By (3.17), the coefficient factors of (3.22) never vanishes, due to $\sigma_3 + \frac{2}{\alpha} = \frac{8}{\alpha(\alpha^2+4)} \neq 0$ and by (3.20), $h - \sigma_3 + \frac{2}{\alpha} = \frac{\alpha^4+14\alpha^2+36}{\alpha(\alpha^2+6)} \neq 0$. Thus, (3.22) is reduced to

$$R(\phi e_{k_j}, e_{k_j})Z = 0. \tag{3.23}$$

By putting $Z = e_{k_j}$ into (3.23) and by using (1.1), the structure tensors ϕ and ϕ_ν are skew-symmetric and $\eta(e_{k_j}) = 0$, we have the following equation.

$$\begin{aligned} 0 &= R(\phi e_{k_j}, e_{k_j})e_{k_j} \\ &= g(e_{k_j}, e_{k_j})\phi e_{k_j} - g(\phi^2 e_{k_j}, e_{k_j})\phi e_{k_j} - 2g(\phi^2 e_{k_j}, e_{k_j})\phi e_{k_j} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\Big\{-g(\phi_\nu\phi e_{k_j},e_{k_j})\phi_\nu e_{k_j}-2g(\phi_\nu\phi e_{k_j},e_{k_j})\phi_\nu e_{k_j}\\
&+g(\phi_\nu\phi e_{k_j},e_{k_j})\phi_\nu\phi^2 e_{k_j}\Big\}+g(Ae_{k_j},e_{k_j})A\phi e_{k_j}\\
&=(4+\lambda_{k_j}\sigma_3)\phi e_{k_j}-4\sum_{\nu=1}^{3}g(\phi_\nu\phi e_{k_j},e_{k_j})\phi_\nu e_{k_j}. \qquad (3.24)
\end{aligned}
$$

Taking the inner product of (3.24) with ϕe_{k_j}, we obtain

$$
\begin{aligned}
0&=(4+\lambda_{k_j}\sigma_3)g(\phi e_{k_j},\phi e_{k_j})-4\sum_{\nu=1}^{3}g(\phi_\nu\phi e_{k_j},e_{k_j})g(\phi_\nu e_{k_j},\phi e_{k_j})\\
&=4\Big(\frac{\alpha^2+5}{\alpha^2+4}\Big)+4\sum_{\nu=1}^{3}g^2(\phi_\nu\phi e_{k_j},e_{k_j}),
\end{aligned}
$$

where we have used $\lambda_{k_j}=-\frac{2}{\alpha}$ and $\sigma_3=\frac{-2\alpha}{\alpha^2+4}$. Since the right side of the equation is greater than 4, this is a contradiction. Thus, Subcase I cannot occur.

Subcase II $\lambda_{k_j}=0$, where $j\geq 2$. Putting $X=e_{k_j}$ into (2.8),

$$0=R_\xi(e_{k_j})=e_{k_j}+2\eta_2(e_{k_j})\xi_2+2\eta_3(e_{k_j})\xi_3-\phi_1\phi e_{k_j}. \qquad (3.25)$$

Applying ϕ_1 (resp., A) to (3.25), we have

$$0=\phi_1 e_{k_j}+2\eta_2(e_{k_j})\xi_3-2\eta_3(e_{k_j})\xi_3+\phi e_{k_j}, \qquad (3.26)$$
$$0=A\phi_1 e_{k_j}+2\eta_2(e_{k_j})A\xi_3-2\eta_3(e_{k_j})A\xi_3+A\phi e_{k_j}. \qquad (3.27)$$

By (2.10), we get

$$A\phi_1 e_{k_j}=0. \qquad (3.28)$$

Putting $X=e_{k_j}$ into (3.16), we obtain

$$0=\frac{\alpha}{2}A\phi e_{k_j}+4\eta_3(e_{k_j})\xi_2-4\eta_2(e_{k_j})\xi_3. \qquad (3.29)$$

Taking the inner product of (3.29) with $\phi_1 e_{k_j}$ and using (3.28), we have $\eta_3^2(e_{k_j})+\eta_2^2(e_{k_j})=0$, that is,

$$\eta_3(e_{k_j})=\eta_2(e_{k_j})=0. \qquad (3.30)$$

By using (3.28) and (3.30), (3.27) becomes

$$A\phi e_{k_j}=0. \qquad (3.31)$$

By (3.26), (3.31) and putting $X = \phi e_{k_j}$ into (2.8), we get

$$R_\xi(\phi e_{k_j}) = \phi e_{k_j} + 2\eta_3(e_{k_j})\xi_2 - 2\eta_2(e_{k_j})\xi_3 + \phi_1 e_{k_j} + \alpha A\phi e_{k_j} = 0.$$

Thus, $\mathfrak{D}_0(x) \ominus [\xi](x)$ is ϕ-invariant. By virtue of (3.10), S has the same eigenvalue $4m+8$ corresponding to each $e_{k_j} \in \mathfrak{D}_0(x) \ominus [\xi](x)$, where $j \geq 2$. Since $S\xi = \sigma\xi$, $\mathfrak{D}_0(x) \ominus [\xi](x)$ is ϕ-invariant and $SX = \sigma X$ for $X \in \mathfrak{D}_1(x)$, we have $S\phi X = \phi SX$ for all tangent vectors X on M. Again, by [15, Theorem], M is locally congruent to a real hypersurface of Type (A).

Now, we verify whether a real hypersurface of Type (A) denoted by M_A satisfies the assumption in our Theorem. We assume that M_A satisfies the condition of Ricci semi-symmetric.

Putting $Y = \xi$ and $Z = \xi$ into (2.1), we have

$$SR_\xi(X) = \sigma R_\xi(X), \tag{3.32}$$

where $\sigma = 4m + h\alpha - \alpha^2$.

In [2, Proposition 3], we obtain the following:

$$SX = \begin{cases} (4m + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\ (4m + 6 + h\beta - \beta^2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\ (4m + 6 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\ (4m+8)X & \text{if } X \in T_\mu, \end{cases}$$

$$R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ (\alpha\beta + 2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\ (\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\ 0 & \text{if } X \in T_\mu, \quad \text{and} \end{cases}$$

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0.$$

Putting $X = \xi_2$ (resp., $X \in T_\lambda$) into (3.32), we have

$$6 + h\beta - \beta^2 = h\alpha - \alpha^2, \tag{3.33}$$

$$6 + h\lambda - \lambda^2 = h\alpha - \alpha^2. \tag{3.34}$$

Combining (3.33) and (3.34), we have $(h - \beta - \lambda)(\beta - \lambda) = 0$. Since $\beta \neq \lambda$, it is

$$h - \beta - \lambda = 0. \tag{3.35}$$

Combining (3.35) and (3.33), we have

$$4 = h\alpha - \alpha^2 = \beta\lambda.$$

This contradicts to the value of β and λ.

Hence, the model space M_A in $G_2(\mathbb{C}^{m+2})$ does not satisfy the Ricci semi-symmetric condition. □

For a Hopf hypersurface with $\xi \in \mathcal{Q}$, by [7, Main Theorem], we know that M is locally congruent to a real hypersurface of Type (B). Now we check whether a Hopf hypersurface of Type (B) denoted by M_B satisfies the Ricci semi-symmetric condition. On TM_B, since $\xi \in \mathcal{Q}$ and $h = \mathrm{Tr}(A) = \alpha + (4n-1)\beta$ is a constant, putting $Y = \xi$ and $Z = \xi$ into (*), we have

$$SR_\xi(X) = \sigma_0 R_\xi(X), \tag{3.36}$$

where $\sigma_0 = 4m + 4 + h\alpha - \alpha^2$.

In [2, Proposition 2], we obtain the following:

$$SX = \begin{cases} (4m+4+h\alpha-\alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\ (4m+4+h\beta-\beta^2)\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ (4m+8)\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (4m+7+h\lambda-\lambda^2)X & \text{if } X \in T_\lambda \\ (4m+7+h\mu-\mu^2)X & \text{if } X \in T_\mu, \end{cases} \tag{3.37}$$

$$R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ \alpha\beta\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ 4\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1+\alpha\lambda)\phi X & \text{if } X \in T_\lambda \\ (1+\alpha\mu)\phi X & \text{if } X \in T_\mu. \end{cases} \tag{3.38}$$

Putting $X = \phi_\ell\xi$ (resp., $X \in T_\lambda$) into (3.36). It gives $h\beta - \beta^2 = 4$ and $h\beta - \beta^2 = -1$. It causes a contradiction.

Remark 3.2 The model space of M_B in $G_2(\mathbb{C}^{m+2})$ does not satisfy the Ricci semi-symmetric condition.

Combining Lemmas 2.1, 3.1 and Remark 3.2, this completes the proof of Theorem in the introduction.

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References

1. Alekseevskii, D.V.: Compact quaternion spaces. Funct. Anal. Appl. **2**, 106–114 (1966)
2. Berndt, J., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians. Mon. Math. **127**, 1–14 (1999)
3. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambrige University Press, Cambrige (1985)
4. Jeong, I., Machado, C.J.G., Pérez, J.D., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with $\mathfrak{D}^\perp$-parallel structure Jacobi operator. Int. J. Math. **22**(5), 655–673 (2011)
5. Jun, J.B., De, U.C., Pathak, G.: On Kenmotsu manifolds. J. Korean Math. Soc. **42**(3), 435–445 (2005)
6. Lee, H., Choi, Y.S., Woo, C.: Hopf hypersurfaces in complex two-plane Grassmannians with Reeb parallel shape operator. Bull. Malays. Math. Soc. **38**, 617–634 (2015)
7. Lee, H., Suh, Y.J.: Real hypersurfaces of Type B in complex two-plane Grassmannians related to the Reeb vector. Bull. Korean Math. Soc. **47**(3), 551–561 (2010)

8. Lee, H., Suh, Y.J., Woo, C.: Real hypersurfaces with commuting Jacobi operators in complex two-plane Grassmannians. Houst. J. Math. **40**(3), 751–766 (2014)
9. Loo, T.-H.: Semi-parallel real hypersurfaces in complex two-plane Grassmannians. Differ. Geom. Appl. **34**, 87–102 (2014)
10. Panagiotidou, K., Tripathi, M.M.: Semi-parallelism of normal Jacobi operator for Hopf hypersurfaces in complex two-plane Grassmannians. Mon. Math. **172**(2), 167–178 (2013)
11. Pak, E., Suh, Y.J., Woo, C.: Real hypersurfaces in complex two-plane Grassmannians with commuting restricted Jacobi operators. Houst. J. Math. **41**(3), 767–783 (2015)
12. Pérez, J.D., Suh, Y.J.: The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians. J. Korean Math. Soc. **44**, 211–235 (2007)
13. Pérez, J.D., Suh, Y.J., Woo, C.: Real hypersurfaces in complex two-plane Grassmannians with GTW Harmonic curvature. Can. Math. Bull. **58**(4), 835–845 (2015)
14. Pérez, J.D., Suh, Y.J., Woo, C.: Real hypersurfaces in complex two-plane Grassmannians whose shape operator is recurrent for the generalized Tanaka-Webster connection. Turk. J. Math. **39**(3), 313–321 (2015)
15. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor. J. Geom. Phys. **60**, 1792–1805 (2010)
16. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor. Proc. R. Soc. Edinb. A. **142**, 1309–1324 (2012)
17. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with Reeb parallel Ricci tensor. J. Geom. Phys. **64**, 1–11 (2013)
18. Suh, Y.J., Hwang, D.H., Woo, C.: Real hypersurfaces in complex two-plane Grassmannians with recurrent Ricci tensor. Int. J. Geom. Methods Mod. Phys. **12**(9), 1550086 (2015)